原　著

Linear Systems Analysis of Cardiovascular System Expressed by an Equivalent Electrical Circuit Model.

— stability, controllability, singular values and system optimization —

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Abstract

Engineering linear systems analysis was applied for cardiovascular system to evaluate an entire systemic properties of the system. The cardiovascular system was expressed by an equivalent electrical circuit comprised of systemic, pulmonary circulations including arterial and venous systems. Linear systems analysis disclosed that the cardiovascular system was stable but uncontrollable. The singular values elevated at low frequency range but was reduced significantly by optimized feed back that minimized the performance function involving the squares of deviations of the target pressures and consumption of control inputs. Present investigation must afford some insights to disclose the characteristic properties of cardiovascular system.

Key Words: Cardiovascular system, Electrical circuit, Linear systems analysis, Optimization.

Introduction

Predicting and evaluating an entire property of circulatory system are particularly important for Anesthesia. Biological experiments[1,2], however, have been performed only to characterize hemodynamical and mechanical aspects of circulatory system. Such efforts could not disclose any total associated feature of circulation as a system such as stability and controllability of cardiovascular system as a whole.

In the present study, we propose an engineering method[3,4] to evaluate the entire behavior of cardiovascular system. We introduce an equivalent electrical circuit model[5] for an entire cardiovascular system including venous system. Applying linear systems analysis, we evaluate stability, controllability and singular values of the cardiovascular system. We show that the cardiovascular system is stable but uncontrollable in engineering sense and show how the system optimization alters the circulatory state.

Method

Mathematical modeling

Fig 1-a shows a schematic illustration of total cardiovascular system (5). It is composed of pulmonary circulation, systemic circulation including abdominal, renal circulation and of venous system. Fig 1-b shows an equivalent electrical circuit model[5] of Fig 1-a. Blood flow in each segment was expressed by pressure Pn and flow fn. Viscous fluid resistance was expressed by resistance R mmHg sec/ml. For the simplicity, we associated the valvular resistance with fluid resistance. Compliance of vessel wall was represented by capacitance C ml/mmHg. Inertial force on the blood is distinct only at near the ventricle and we set inertance L mmHg sec^2/ml only at aorta and pulmonary artery. We also set the resistance between the right and left atriums for atrial septum defect and

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Fig 1-a

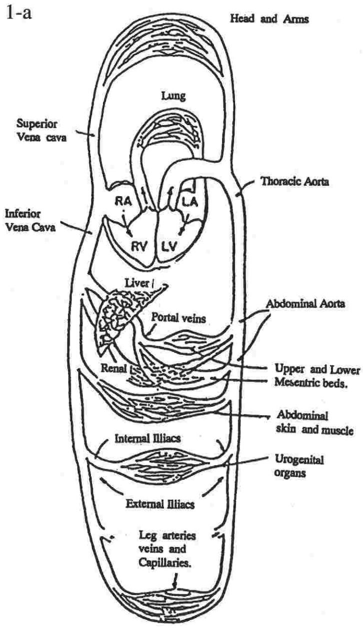

Fig 1-b

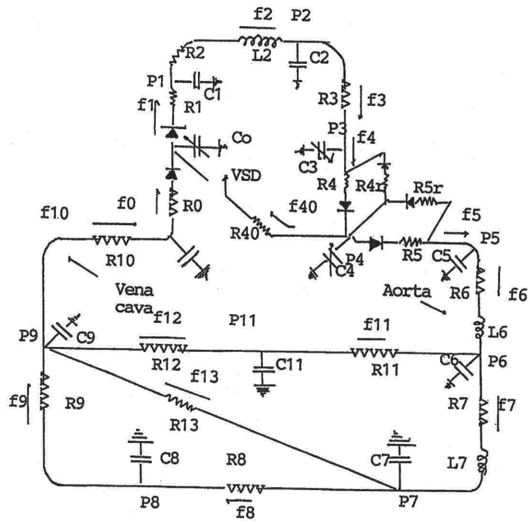

Fig. 1 Modeling of the cardiovascular system[5)].
Fig1-a. Schematic illustration of the total cardiovascular system. Fig 1-b. An equivalent electrical circuit of Fig 1-a.

the resistance between the ventricles for ventricular septum defect. Abbreviations of Fig 1-b are listed in Table 1. Pressure (Pn)- flow (fn) relations of viscous fluid in a cylindrical segment can be approximated by the Ohmic law

$$f_n(t) - f_{n+1}(t) = C_n \, dP_n/dt \quad \cdots\cdots (1)$$

$$P_n(t) - P_{n+1}(t) = L_{n+1} \, df_{n+1}(t)/dt + R_{n+1} f_{n+1}(t) \quad \cdots\cdots (2)$$

P0(t) and P4(t) were set as controlling inputs generated in right ventricle and left ventricle. The differential equations for pressures were

$$dP_1(t)/dt = (f_1(t) - f_2(t))/C1 \quad \cdots\cdots (3)$$

$$dP_2(t)/dt = (f_2(t) - f_3(t))/C2 \quad \cdots\cdots (4)$$

$$dP_3(t)/dt = (f_3(t) - f_4(t))/C3 \quad \cdots\cdots (5)$$

$$dP_5(t)/dt = (f_5(t) - f_6(t))/C5 \quad \cdots\cdots (6)$$

$$dP_6(t)/dt = (f_6(t) - f_{11}(t) - f_7(t))/C6 \quad \cdots\cdots (7)$$

$$dP_7(t)/dt = (f_7(t) - f_8(t) - f_{13}(t))/C7 \quad \cdots\cdots (8)$$

$$dP_8(t)/dt = (f_8(t) - f_9(t))/C8 \quad \cdots\cdots (9)$$

$$dP_9(t)/dt = (f_9(t) - f_{10}(t) + f_{12}(t) + f_{13}(t))/C9 \quad \cdots (10)$$

$$dP_{10}(t)/dt = (f_{10}(t) - f_0(t))/C10 \cdots\cdots (11)$$

Table 1 abbreviations for Fig 1-b.

1. Segmental flows
   - f0 : flow from right atrium to right ventricle.
   - f1 : flow from right ventricle to pulmonary artery through pulmonary arterial valve.
   - f2 : flow at pulmonary capillary.
   - f3 : flow in pulmonary vein.
   - f4 : flow from left atrium to left ventricle.
   - f5 : aortic flow.
   - f6 : flow from descending aorta to abdominal aorta.
   - f7 : flow at femoral artery.
   - f8 : flow at systemic capillary artery.
   - f9 : flow at peripheral venous.
   - f10 : flow at vena cava.
   - f11 : flow into kidney.
   - f12 : flow at venous side of abdominal organs.
   - f13 : flow branched from femoral artery perfusing low limbs.
2. Segmental pressures
   - P0 : Right ventricular pressure.
   - P1 : Pulmonary arterial pressure.
   - P2 : Pulmonary capillary pressure.
   - P3 : Pulmonary vein pressure.
   - P4 : Left ventricular pressure.
   - P5 : Aortic pressure.
   - P6 : Abdominal arterial pressure.
   - P7 : Femoral arterial pressure.
   - P8 : Capirally arterial pressure in systemic circulation.
   - P9 : Venous pressure at systemic circulation.
   - P10 : Pressure of right atrium.
   - P11 : Renal arterial pressure.

Table 2 The system parameters.

Resistance (mmHg/ml sec)		Capacitance (ml/mmHg)		Inertance (mmHg/ml sec^2)
R0 = 0.1	R7 = 1.5	C0 = 2	C7 = 0.25	L2 = 0.01
R1 = 0.1	R8 = 2.75	C1 = 0.3	C8 = 0.1	L6 = 0.05
R2 = 0.5	R9 = 0.5	C2 = 0.3	C9 = 0.2	L7 = 0.02
R3 = 0.5	R10 = 0.1	C3 = 1.0	C10 = 0.1	
R4 = 0.1	R11 = 0.5	C4 = 5.0	C11 = 0.1	
R5 = 0.1	R12 = 0.5	C5 = 1.0		
R6 = 1.0	R13 = 0.5	C6 = 0.5		
RASD = 100				
RVSD = 100				

$$dP_{11}(t)/dt=(f_{11}(t)-f_{12}(t))/C11 \quad \cdots\cdots(12)$$

Equations for flow are given in Appendix 1. The standard values[5] of Rn, Cn and Ln were shown in Table 2.

Linear systems analysis

The relations between the pressure and flow are expressed by the Ohmic law. They are given in APPENDIX I. Substituting these equations to the right sides of the equations of the pressures (3) to (12), we can have differential equations only for the pressures[6]. These system equations are expressed in a vector form

$$X'=A\,X+B\,U, \qquad Y=C\,X \quad \cdots\cdots(13)$$

where X is a matrix of the state variables for pressures Pn(t)[6]. A is a matrix that characterizes intrinsic properties of the system which elements are functions of Rn, Cn and Ln. B is a matrix for the control inputs U of the system dedicated from P0(t) and P4(t). C is a matrix for the out puts Y of the system. For the simplicity, we set C unity. Since the circulatory system is a closed system, we can obtain a Laplace transformed transfer function G(s) (where s is Laplace operator)[3] of the system by computer approach though the functional form is too complicated to show.

The stability of the system was evaluated by the eigen values of the characteristic equations[3]. A brief sketch for obtaining the characteristic equations and eigen value by using the state transition function is given in APPENDIX II. By setting $s=j\omega$ (where $j^2=-1$ and ω is angular velocity), we can compute frequency character of the transfer function G(s). We show a method of Bode plot to evaluate the frequency dependency of the transfer function G(s) of the present system. We show gains dB[3] of the present system by computing the absolute value of G(s) and phases degree[3] by computing $\angle G(j\omega)$ as functions of frequency (rad/sec).

We evaluated the system performance of controllability from the stand point of control engineering. The rigorous definition of the controllability is "The state x(t) is controllable at $t=t_o$ when the state can be transferred to any final state $x(t_f)$ for a finite time interval (t_f-t_o) by a continuous input u(t)"[3,7,8].

The controllability[3,4] of the system was judged by the rank of matrix. Precise mathematical process should be referred to Engineering text books[3,7,8] and we give the necessary conditions for the controllability in Appendix III. The number of uncontrollable variables are calculated[3,7,8] by

length (A) − rank (Co) where

$$Co=[B\ AB\ A^2B\ A^3B \quad A^{n-1}B]:n=13 \cdots\cdots(14)$$

We can conclude that the system is controllable only if there was no uncontrollable variable other wise, we judge the system uncontrollable[3,7,8].

The singular values dB of the system[3] for a measure of functional space of the system

$$C(j\omega I-A)^{-1}B \quad \cdots\cdots(15)$$

was calculated as a function of frequency (rad / sec). The criterion of the optimality[9] in engineering sense of the cardiovascular system was expressed by the performance function[4,7]

$$J = \int_0^t [\, X Q X + U R U \,]\, dt \quad \cdots\cdots (16)$$

where Q and R are weighting matrixes for X and for U respectively. The optimal feed back is given by U= −KX where K is the optimal feed back gain that minimizes the performance function. This feed back law implies that P0(t) and P4(t) that are set as controlling inputs U are fed by the venous return which produces finite pressure X. The significance of the performance function has been discussed in our previous work[10].

Results

1. The stability and controllability of the system.

All the real parts of eigen values (Table 3) of the system were negative and insensitive to 100 fold changes in Rn, Cn and Ln. Thus, the system was judged to be stable. Fig 2 shows typical examples of the Bode plot exhibition of the system. Fig 2-a is those of Pulmonary venous pressure and Fig 2-b is those of pulmonary capillary flow. In both cases the Gains (dB) are negative at the cross over frequency 180 deg (denoted by vertical bars on the phases. They show frequency character of stability of these segments. By the matrix calculation, the system was judged to be uncontrollable. There were 7 uncontrollable variables in the system which were insensitive to regulation of the system parameters. Thus, we concluded that the cardiovascular system was uncontrollable in purely engineering definition[3].

2. The singular values with optimized feed back.

Fig 3 shows the singular vales of the system when the system was not supplied the optimized input (Fig 3-a) and those when the system was fed back with the optimized feed back (Fig 3-b) that minimized the performance function. The singular values were depressed considerably when the feed back was optimized.

Table 3 EigenValues and damping

Eigenvalue	Damping	Freq. (rad/s)
−3.35e+00	1.00e+00	3.35e+00
−1.02e+01	1.00e+00	1.02e+01
−1.58e+01	1.00e+00	1.58e+01
−2.02e+01	1.00e+00	2.02e+01
−3.56e+01	1.00e+00	3.56e+01
−5.84e+01+5.38e+01i	7.36e−01	7.94e+01
−5.84e+01−5.38e+01i	7.36e−01	7.94e+01
−6.16e+01	1.00e+00	6.16e+01
−6.97e+01	1.00e+00	6.97e+01
−9.74e+01	1.00e+00	9.74e+01
−1.32e+02	1.00e+00	1.32e+02
−2.33e+02	1.00e+00	2.33e+02
−3.22e+02	1.00e+00	3.22e+02

Fig. 2-a

Fig. 2-b

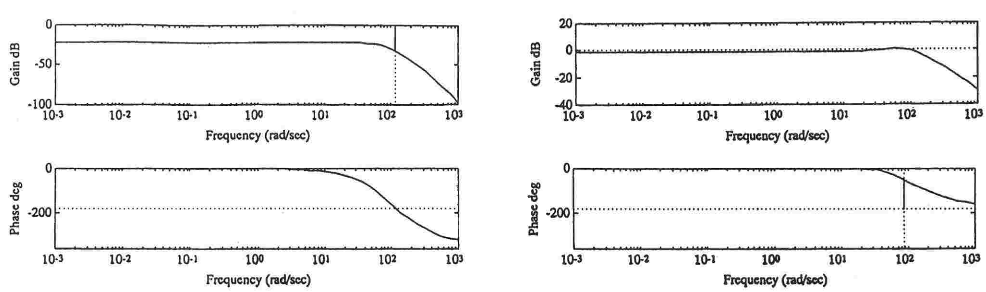

Fig. 2 Bode plot of cardiovascular system.
Fig 2-a is bode plots of Pulmonary venous pressure and Fig 2-b is those of Pulmonary capillary flow. Upper is gain (dB) and lower is phase (deg).

Fig. 3-a

Fig. 3-b

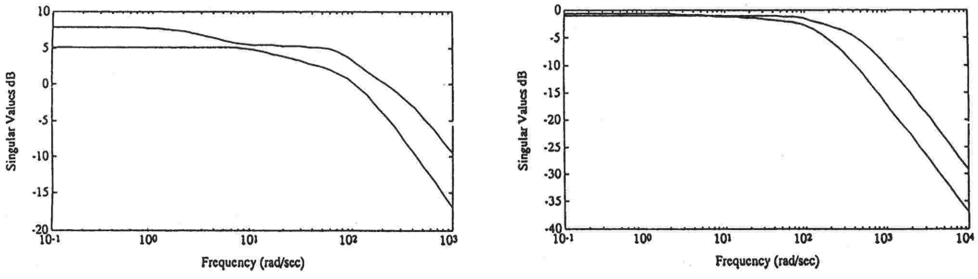

Fig. 3 The singular vales of the system.
when the system was not supplied by the optimized input (Fig 3-a) and when the system was supplied by the optimal feed back which is given by U = −K X where K is the optimal feed back gain that minimizes the performance function. equation (16).

Discussion

To evaluate an entire property of cardiovascular system, we introduced engineering linear system analysis with the help of modeling the system by an equivalent electrical circuit. The cardiovascular system was computed to be stable but uncontrollable. We introduced the singular values so as to measure the functional space of the cardiovascular system. System optimization reduced the singular values significantly. The present method will be available for evaluating the associated features of the cardiovascular system.

Some assumptions were required for modeling the cardiovascular system. An electrical circuit modeling requires the validity of applying the Ohmic law for pressure flow relations. This can be seen by successful simulation of pressure curves by similar electrical circuit model[11]. All the biological systems are non linear and applying the linear systems analysis requires simplification of the biological properties of the system. The present approach is available only for the time invariant, linear system. The physiological performances of the biological systems, however, are almost within a linear range with a little deviation from the normal physiological range[10]. Inherent non linearity of the system can be conquested by dividing each segment piese wisely in detail[11]. Non linear modeling of the system, although it can describe minuet structural properties of the system, will be too complicated to construct and can not be available for actual clinical using. Exogenous control inputs such as autonomic nervous system or humoral factors can be represented by adding more inputs Un on the system equations.

Actual biological cardiovascular systems involve many other factors such as humoral control and auto regulation of arterial beds. These factors concern on the regulation on molecular level and on local circulation. The present investigation can not describe such minute mechanisms, however referred to characterize entire and integrated properties of the system but did not consider these elemental contributions. More precise and complicated mathematical treatment should be required to incorporate such elemental factors.

We introduced engineering criterions so as to introduce a computer linked on-line evaluation system to human cardiovascular system in actual clinical circumstance. The stability of the cardiovascular system is consistent with the physiological reaction of the system against the exogenous disturbance so as to return to the steady stable blood pressure. Among what we have introduced, particularly the uncontrollability of the system may evoke criticism. This conclusion, however derives in purely engineering sense. Rigorously, engineering controllability means that the system can be transferred to any arbitrary state within a finite time[3,7,8]. Actually, it is clear that human cardiovascular system can not be transferred to any arbitrary state. Thus uncontrollability of the cardiovascular system in not inconsistent to its physiological operation where the pressure and flow are maintained within the physiological range.

The singular value of the system is an extension of Bode plot to multiple input multiple output system. Reduction of the singular value by the optimization of the system indicates that the singular value is a measure effective functional performance of the cardiovascular system. Criterion of the system optimization is yet unsettled problem. We introduced the present performance function because it is the standard form[3,4] and can describe the physiological state where the system is organized to minimize the disturbances of the system.

We only introduced a method to evaluate entire properties of the cardiovascular system by an on line computational system that should be linked to Anesthesia. Further improvement of the present method will be required for clinical application.

Conclusion

Cardiovascular system is stable but uncontrollable. Optimized feed back that minimizes the deviations from the physiological state reduces the singular values. Linear systems analysis is available for evaluating entire properties of the cardiovascular system.

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Appendix I. The equations and differential equations for flow rates are

$$f_0(t)=(P_{10}(t)-P_0(t))/R0$$
$$f_1(t)=(P_0(t)-P_1(t))/R1$$
$$df_2(t)/dt=(P_1(t)-P_2(t)-R2\, f_2(t))/L2$$
$$f_3(t)=(P_2(t)-P_3(t))/R3$$
$$f_4(t)=(P_3(t)-P_4(t))/R4$$
$$f_5(t)=(P_4(t)-P_5(t))/R5$$
$$d\, f_6(t)/dt=(P_5(t)-P_6(t)-R6\, f_6(t))/L6$$
$$d\, f_7(t)/dt=(P_6(t)-P_7(t)-R7\, f_7(t))/L7$$
$$f_8(t)=(P_7(t)-P_8(t))/R8$$
$$f_9(t)=(P_8(t)-P_9(t))/R9$$
$$f_{10}(t)=(P_9(t)-P_{10}(t))/R10$$
$$f_{11}(t)=(P_6(t)-P_{11}(t))/R11$$
$$f_{12}(t)=(P_{11}(t)-P_9(t))/R12$$
$$f_{13}(t)=(P_7(t)-P_9(t))/R13$$

Appendix II. Laplace transform of the system and its characteristic equation by the state transition matrix ϕ (t) of the system.

The state transition matrix is defined as a matrix that satisfies the linear homogeneous state equation

$$X(t)'=A\,X(t) \qquad \cdots(1)$$

Let ϕ (t) be an n and n matrix that represents the state transition matrix. then it must satisfy the equation

$$\phi(t)'=A\,\phi(t) \qquad \cdots(2)$$

Let X(0) denote the transient state of the system at t= 0 ; then ϕ (t) is also defined by the matrix equation

$$X(t)=\phi(t)\,X(0) \qquad \cdots(3)$$

which is the solution of the homogeneous state equation for $t > 0$. Taking the Laplace integral transformation on both sides of Eq (3), we have

$$s\,X(s)-X(0)=A\,X(s) \quad \cdots\cdots\cdots(4)$$

where s is the Laplace operator. Solving for X(s)

$$X(s)=(s\,I-A)^{-1}\,X(0) \cdots\cdots\cdots(5)$$

Taking the inverse Laplace transform L^{-1} on both sides of the Eq(5)

$$X(t)=L^{-1}\,[\,(s\,I-A)^{-1}\,]\,X(0) \quad \cdots\cdots\cdots(6)$$

Comparing to Eq(3) with Eq(6), the state transition matrix is identified to be

$$\phi\,(t)=L^{-1}\,[\,(s\,I-A)^{-1}\,] \quad \cdots\cdots\cdots(7)$$

where sI - A is the characteristic equation and its roots are the eigen values of the system.

Appendix III. The necessary condition for the controllability.

When the system is controllable in which the system can be transfered from any arbitrary state x(0) to any desired state x(tf) within a finite time duration tf. Then, by utilyzing the state transition function ϕ (t)=exp (A t), we have

$$x(t)=e^{At}\,x(0)+\int_0^t e^{A(t-\tau)}\,B\,u(\tau)\,d\tau$$

For an arbitrary x(0), there is an input $u(\tau)$, $0<\tau<t$ and following relation can be hold.

$$0=e^{At}\,x(0)+\int_0^t e^{A(t-\tau)}\,B\,u(\tau)\,d\tau$$

$$=e^{At}\,[\,x(0)+\int_0^t e^{-A\tau}\,B\,u(\tau)\,d\tau\,]=0$$

Since e^{At} is non singular matrix , contents within [ ]= 0.

$$x(0)=-\int_0^t e^{-A\tau}\,B\,u(\tau)\,d\tau$$

Because e^{At} is atransition matrix, it is expanded as

$$e^{At}=I+At+A^2t^2/2!+****=\sum_{i=0}^{00} A^i t^i/i!$$

By the theorem of Kaley Hamilton[3], above e^{At} can be expanded by finite series

$$=\alpha_0(t)\,I+\alpha_1(t)\,A+***\alpha_{n-1}(t)\,A^{n-1}=\sum_{i=0}^{n-1}\alpha_i(t)\,A^i$$

Therefore, substituting this series to e^{At}

$$x(0)=-\int_0^t e^{-A\tau}B\,u(\tau)\,d\tau=\sum_{i=0}^{n-1}[\,A^i\,B\int_0^t \alpha_i(\tau)\,u(\tau)\,d\tau\,]$$

$$=[\,B\;AB\;A^2B\;A^3B\;A^{n-1}B\,]\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}$$

Here

$$u_i=-\int_0^t \alpha_i(\tau)\,u(\tau)\,d\tau$$

To hold above expression for an arbitrary x(0),it is evident that there should be n independent factors in the matrix

$$Q=[\,B\;AB\;A^2B\;A^3B\;A^{n-1}B\,]$$

Therefore rank Q=n.

(Circ Cont 20 : 163~169, 1999)